M.Sc. (Mathematics) (NEP Pattern) Semester-I

NEP-61 - DSC-1 - Advanced Abstract Algebra

P. Pages : 2 **GUG/S/25/15112**

Time : Three Hours * 2 7 5 6 * Max. Marks : 80

Notes : 1. Solve all **five** questions.
2. All questions carry equal marks.

1. a) Prove that for a group G any nonempty subset S of G, N(S) is a subgroup of G. Further prove that, for any subgroup H of G. **8**

i) N(H) is the largest subgroup of G in which H is normal.

ii) If K is a subgroup of N(H), then H is a normal subgroup of KH.

b) State and prove second isomorphism theorem. **8**

OR

c) Show that a symmetric group S_3 has a trivial centre {e}. Also show that $\ln(S_3) \cong S_3$. **8**

d) State and prove Burnside theorem. **8**

2. a) Prove that: If G is solvable group, then every subgroup of G and every homomorphic image of G are solvable. Conversely, if N is a normal subgroup of G such that N and G/N are solvable, then G is solvable. **8**

b) Prove that every finite group has a composition series. **8**

OR

c) Prove that: If $\alpha, \sigma \in S_n$, then $\tau = \alpha \sigma \alpha^{-1}$ is the permutation obtained by applying α to the symbols in σ. Hence, any two conjugate permutations in S_n have the same cycle structure Conversely, any two permutations in S_n with the same cycle structure are conjugate. **8**

d) Prove that : if a permutation $\sigma \in S_n$ is a product of r transportations and also a product of s transposition, then r and s are either both even or both odd. **8**

3. a) Let G be a finite group, and let p be a prime. If p^m divides $|G|$, then prove that G has a subgroup of order p^m. **8**

b) Show that: if each element $\neq e$ of a finite group G is of order 2, then $|G| = 2^n$ and $G \cong C_1 \times C_2 ... \times C_n$ where C_i are a cyclic and $|C_i| = 2$. **8**

OR

c) If the order of a group is 42, prove that its Sylow 7-subgroup is normal. **8**

d) Let G be a group of order 108. Show that there exists a normal subgroup of order 27 or 9. **8**

4. a) Let $f: R \to S$ be a homomorphism of a ring R into a ring S. Then prove that: **8**

i) If 0 is the zero of R, then f(0) is the zero of S.

ii) If $a \in R$, then $f(-a) = -f(a)$.

iii) $f(R)$ is a subring of S.

iv) $f^{-1}(0)$ is an ideal in R.

b) Prove that: In a nonzero commutative ring with unity, an ideal M is maximal if and only if R/M is field. **8**

OR

c) Let R be a Boolean ring, Then show that each prime ideal $P \neq R$ is maximal. **8**

b) If R is nonzero a ring with unity 1, and I is an ideal in R such that $I \neq R$, then prove that there exists a maximal ideal M of R such that $I \subseteq M$. **8**

5. a) Define: **4**

i) Maximal normal subgroup.

ii) Simple group.

b) If a cyclic group has exactly one composition series, then show that it is a p-group. **4**

c) Prove that: If G is a cyclic group of order mn, where (m, n) = 1, then $G \cong H \times K$, where H is a subgroup of order m and K is a subgroup of order n. **4**

d) Define: **4**

i) Principal ideal

ii) Maximal ideal

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